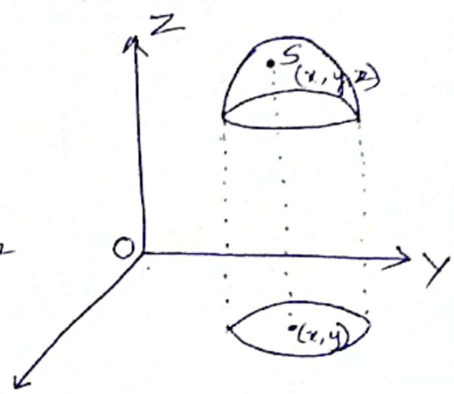
Computation of Surface area (Using double Integral)

The double integral can be made use in evaluating the surface area of a surface.



Consider a surface 'S' in Space. Let the equation of the surface S be $Z=f(x,y)$. It can be proved that the surface area of the surface is given by

$$S=\iint_A \left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dx\,dy$$

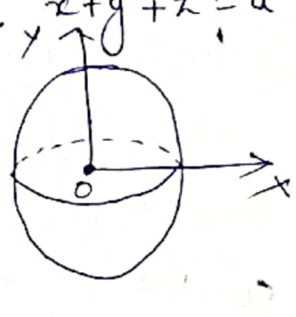
, where 'A' is the region representing the projection of S on the xy-plane. Note that (x,y) vary over 'A' as (x,y,z) vary over S.

IIIly if B and C are the projections of 'S' on the yz-plane and zx-plane respectively, then

$$S=\iint_B \left[1+\left(\frac{\partial x}{\partial y}\right)^2+\left(\frac{\partial x}{\partial z}\right)^2\right]^{1/2} dy\,dz \quad \text{and} \quad S=\iint_C \left[1+\left(\frac{\partial y}{\partial z}\right)^2+\left(\frac{\partial y}{\partial x}\right)^2\right]^{1/2} dz\,dx$$

Problems

1. Find the Surface area of the sphere, $x^2+y^2+z^2=a^2$



Soln The surface area is twice the surface area of the upper part of the given sphere, $z=(a^2-x^2-y^2)^{1/2}$, $(z>0)$

$$\frac{\partial z}{\partial x}=\frac{1}{2}\left(a^2-x^2-y^2\right)^{-1/2}(-2x)=\frac{-x}{(a^2-x^2-y^2)^{1/2}}$$

$$\frac{\partial z}{\partial y}=\frac{-y}{(a^2-x^2-y^2)^{1/2}}$$

$$\therefore \quad 1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = \frac{a^2}{a^2-x^2-y^2}$$

Hence, the Surface area is

$$S = 2\iint_A \left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dx\,dy = 2\iint_A \left\{\frac{a^2}{(a^2-x^2-y^2)}\right\}^{1/2} dx\,dy$$

Now, 'A' is the projection of the sphere on the xy-plane. We note that this projection is the area bounded by the circle $x^2+y^2=a^2$. Hence, in A, θ varies from 0 to 2π and 'r' varies from 0 to a, where (r,θ) are the polar coordinates. Put $x=r\cos\theta$, $y=r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$$\therefore \quad S = 2\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{a}{\sqrt{a^2-r^2}}\, r\,dr\,d\theta = 2a\int_0^{2\pi} d\theta \times \int_0^a \frac{r}{\sqrt{a^2-r^2}}\,dr$$

$$= 2a\int_0^{2\pi} d\theta \left\{-\sqrt{a^2-r^2}\right\}_0^a = 2a\int_0^{2\pi} d\theta\,\{a\} = 2a^2\left[\theta\right]_0^{2\pi} = 4\pi a^2.$$

2. Find the surface area of the portion of the cylinder $x^2+z^2=a^2$, which lies inside the cylinder $x^2+y^2=a^2$

Sol^n. Let S_1 be the cylinder $x^2+z^2=a^2$ and S_2 be the cylinder $x^2+y^2=a^2$.

For the cylinder S_1: $\frac{\partial z}{\partial x} = -\frac{x}{z}$, $\frac{\partial z}{\partial y} = 0$

So that, $1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = 1+\frac{x^2}{z^2}+0 = \frac{z^2+x^2}{z^2} = \frac{a^2}{a^2-x^2}$

The required surface area is twice the surface area of the upper part of the cylinder S_1 which lies inside the cylinder $x^2+y^2=a^2$. Hence, the required surface area is

$$S = 2\iint_A \left[1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]^{1/2} dA = 2\iint_A \frac{a}{\sqrt{a^2-x^2}}\,dA$$

where 'A' is the projection of the cylinder S_1 on the xy plane that lies within the cylinder $S_2 : x^2+y^2=a^2$. In A, 'x' varies from $-a$ to a and for each x, 'y' varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$.

$$\therefore \quad S = 2\int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{a}{\sqrt{a^2-x^2}}\,dy\,dx$$

$$= 2a\int_{-a}^{a} \frac{1}{\sqrt{a^2-x^2}}\Big[y\Big]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx = 2a\int_{-a}^{a}\frac{1}{\sqrt{a^2-x^2}}\left[2\sqrt{a^2-x^2}\right]dx$$

$$= 4a\int_{-a}^{a} dx = 4a\Big[x\Big]_{-a}^{a} = 8a^2$$

3. Find the surface area of the portion of the sphere $x^2+y^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.

Sol^n $z = a^2-x^2-y^2 \Rightarrow 2z\frac{\partial z}{\partial x} = -2x \Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}$ and $\frac{\partial z}{\partial y} = \frac{-y}{z}$

$$\therefore \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}} = \sqrt{\frac{z^2+x^2+y^2}{z^2}}$$

$$= \sqrt{\frac{a^2}{z^2}} = \frac{a}{z} = \frac{a}{\sqrt{a^2-x^2-y^2}}$$

$\therefore$ Required surface area $= S = 4\iint_R \frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$

changing to polar coordinates, we get

$$S = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sin\theta} \frac{a}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta = 4a\int_0^{\pi/2}\left[\int_0^{a\sin\theta}\frac{r}{\sqrt{a^2-r^2}}\,dr\right]d\theta$$

$$= 4a\int_{\theta=0}^{\pi/2}\left[-\sqrt{a^2-r^2}\right]_0^{a\sin\theta} d\theta = -4a\int_0^{\pi/2} a(\cos\theta-1)\,d\theta = 4a^2\int_0^{\pi/2}(1-\cos\theta)\,d\theta$$

$$= 4a^2\left[\theta-\sin\theta\right]_0^{\pi/2} = 4a^2\left[\frac{\pi}{2}-1\right] = 2a^2(\pi-2)$$

4) Find the area of the Surface $az = xy$ that lies inside the Cylinder $(x^2+y^2)^2 = 2a^2xy$

Sol^n We have $az = xy$

$$\therefore \frac{\partial z}{\partial x} = \frac{y}{a}, \quad \frac{\partial z}{\partial y} = \frac{x}{a}$$

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{x^2+y^2+a^2}{a^2}} = \frac{1}{a}\sqrt{a^2+r^2} \quad \left[\text{Changing in polars}\right]$$

Also, $(x^2+y^2)^2 = 2a^2xy$, becomes $r^2 = a^2\sin 2\theta$

$$\therefore S = 2\int_0^{\pi/2}\int_0^{a\sqrt{\sin 2\theta}} \frac{\sqrt{a^2+r^2}}{a} \times r\,d\theta\,dr = \frac{2}{a}\int_0^{\pi/2}\left[\frac{(a^2+r^2)^{3/2}}{2\times 3/2}\right]_0^{a\sqrt{\sin 2\theta}} d\theta$$

$$= \frac{2}{3}a^2\int_0^{\pi/2}\left[(1+\sin 2\theta)^{3/2}-1\right]d\theta = \frac{2}{3}a^2\int_0^{\pi/2}\left[(\sin\theta+\cos\theta)^3-1\right]d\theta = \frac{1}{9}(20-3\pi)a^2$$

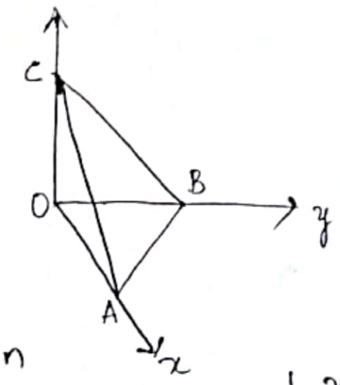
Volume Underneath a Surface

$$V = \iint_A f(x,y)\,dx\,dy = \iint_A z\,dx\,dy$$

Problems

1\. Find the volume of the tetrahedron bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

sol^n Required Volume is bounded by the Coordinate planes & the plane ABC whose equation is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ or



$z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$.

The projection of the plane ABC on the xy plane is the triangular area OAB. In this area, x Varies from $x=0$ to $x=a$ and for each x, y Varies from '0' to a point on the line AB whose equation is $\frac{x}{a}+\frac{y}{b}=1$, So that $y=b\left(1-\frac{x}{a}\right)$ on AB.

Hence, the required volume is

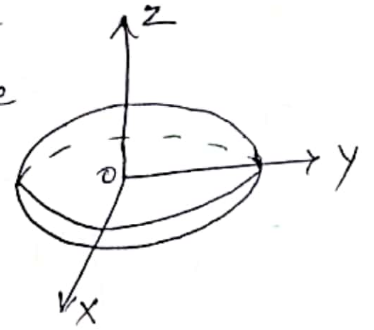
$$V=\int_{OAB} Z\,dA=\int_{x=0}^{a}\int_{y=0}^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=c\int_0^a\left\{y-\frac{xy}{a}-\frac{y^2}{2b}\right\}_0^{b(1-x/a)}dx=c\int_0^a\left(\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right)dx$$

$$=\frac{bc}{2}\left[-\frac{a}{3}\left(1-\frac{x}{a}\right)^3\right]_0^a=\frac{1}{6}abc$$

2\. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol^n 'S' denotes the Surface of the ellipsoid above the xy-plane. The equation of this surface is



$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ (z>0)$ (or)

$$z=c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{1/2}=f(x,y)$$

The volume of the region bounded by the surface and the xy-plane gives the volume V_1 of the upper half of the full ellipsoid. The volume

$$V_1 = \iint_A f(x,y)\,dx\,dy$$, A = area of the projection of 'S' on the xy plane.

A is the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore V_1 = c \iint_A \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2} dx\,dy = c\left(\frac{2}{3}\pi a b\right) = \frac{2}{3}\pi abc$$

Volume of full ellipsoid is $2V_1$.

3) Find the volume bounded by the paraboloid $x^2+y^2=az$, the cylinder $x^2+y^2=2ay$ and the plane $z=0$.

Sol^n. Volume beneath the surface $z = \frac{x^2+y^2}{a}$ and bounded by the cylinder $x^2+y^2=2ay$. The projection of this surface on the xy plane is the circular area A bounded by the circle $x^2+y^2=2ay$, the polar equation of which $r = 2a\sin\theta$. In this circular area θ varies from 0 to π, & for each θ, 'r' varies from 0 to $2a\sin\theta$.

$$V = \iint_A z\,dx\,dy = \iint_A \left(\frac{x^2+y^2}{a}\right) dx\,dy = \int_{\theta=0}^{\pi} \int_{r=0}^{2a\sin\theta} \left(\frac{r^2}{a}\right) r\,dr\,d\theta.$$

$$= \frac{1}{a}\int_0^{\pi} \left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta = 4a^3 \int_0^{\pi} \sin^4\theta\,d\theta = 8a^3 \int_0^{\pi/4} \sin^4\theta\,d\theta$$

$$= 8a^3 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{3\pi a^3}{2}$$

Volume of Revolution Using double Integrals (4)

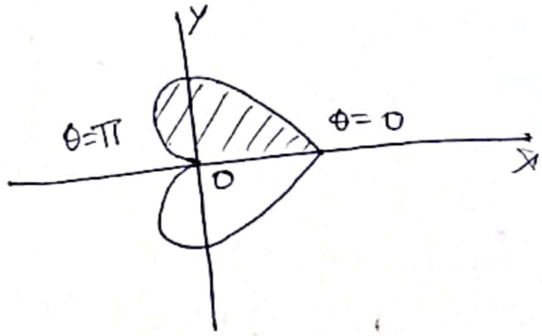
Let $y=f(x)$ be a simple plane curve enclosing an area A. Suppose this curve is revolved about x-axis. Then it can be proved that the volume of the solid generated is given by

$$V=\iint_A 2\pi y\, dA = \iint_A 2\pi y\, dx\, dy$$

In polar form, this formula becomes $V=\iint_A 2\pi r^2 \sin\theta\, d\theta\, dr$

Problems

1. Find the volume generated by the revolution of the cardiod $r=a(1+\cos\theta)$ about the initial line



Sol^n 'θ' varies from 0 to π.

'r' varies from 0 to $a(1+\cos\theta)$

$$V=\int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta\, dr\, d\theta$$

$$= 2\pi\int_0^{\pi} \sin\theta \left\{\left[\frac{r^3}{3}\right]_0^{a(1+\cos\theta)}\right\} d\theta$$

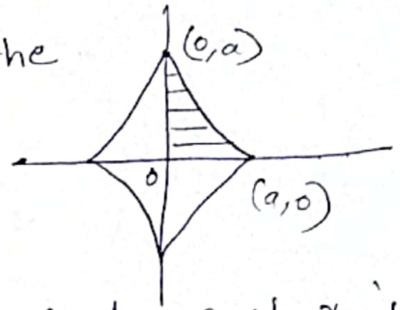
$$= \frac{2\pi a^3}{3}\int_0^{\pi} (1+\cos\theta)^3 \sin\theta\, d\theta = \frac{2\pi a^3}{3}\left[-\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= \frac{8}{3}\pi a^3$$

2). Find the volume of the solid generated by revolving the asteroid $x^{2/3}+y^{2/3}=a^{2/3}$ about x-axis.

Soln

$$V = 2\iint_A 2\pi y\,dx\,dy$$, A is the area bounded by the curve in I quadrant.



In A, 'x' varies from 0 to a & for each x, 'y' varies from 0 to $\left(a^{2/3}-x^{2/3}\right)^{3/2}$.

$\therefore$

$$V = 4\pi\int_{x=0}^{a}\int_{y=0}^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} y\,dy\,dx = 4\pi\int_0^a\left(\frac{y^2}{2}\right)_0^{\left(a^{2/3}-x^{2/3}\right)^{3/2}} dx$$

$$= 2\pi\int_0^a\left(a^{2/3}-x^{2/3}\right)^{3/2} dx.$$

Put $x = a\cos^3 t$

$dx = -3a\cos^2 t\sin t\,dt$

$$V = 2\pi\int_{\pi/2}^{0}\left(a^{2/3}-a^{2/3}\cos^2 t\right)^3\left(-3a\cos^2 t\sin t\,dt\right).$$

$$= 6\pi a^3\int_0^{\pi/2}\sin^7 t\cos^2 t\cdot dt = 6\pi a^3\,\frac{6\cdot4\cdot2}{9\cdot7\cdot5\cdot3} = \frac{32}{105}\pi a^3$$